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Non-commutative solitons and strong-weak duality

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Abstract: Some properties of the non-commutative versions of the sine-Gordon model (NCSG) and the corresponding massive Thirring theories (NCMT) are studied. Our method relies on the NC extension of integrable models and the master Lagrangian approach to deal with dual theories. The master lagrangians turn out to be the NC versions of the so-called affine Toda model coupled to matter fields (NCATM) associated to the group GL(2), in which the Toda field belongs to certain representations of either $U(1) \times U(1)$ or $U(1)_C$ corresponding to the Lechtenfeld et al. ($NCSG_1$) or Grisaru-Penati ($NCSG_2$) proposals for the NC versions of the sine-Gordon model, respectively. Besides, the relevant $NCMT_{1,2}$ models are written for two (four) types of Dirac fields corresponding to the Moyal product extension of one (two) copy(ies) of the ordinary massive Thirring model. The $NCATM_{1,2}$ models share the same one-soliton (real Toda field sector of model 2) exact solutions, which are found without expansion in the NC parameter θ for the corresponding Toda and matter fields describing the strong-weak phases, respectively. The correspondence $NCSG_1 \leftrightarrow NCMT_1$ is promising since it is expected to hold on the quantum level.

Keywords: Integrable Hierarchies, Non-Commutative Geometry, Integrable Equations in Physics, Integrable Field Theories.

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JHEP03(2005)037

Contents

1. Introduction

Field theories in non-commutative (NC) space-times are receiving considerable attention in recent years in connection to the low-energy dynamics of D-branes in the presence of background B-field (see, e.g. refs. [1]). In particular, the NC versions of integrable systems (in two dimensions) are being considered (see e.g. [2]). It is believed that these models, defined on two-dimensional NC *Euclidean* space, turn out to be the NC versions of statistical models in the critical points and in the off-critical integrable directions.

Some non-commutative versions of the sine-Gordon model (NCSG) have been proposed in the literature [3]–[6]. The relevant equations of motion have the general property of reproducing the ordinary sine-Gordon equation when the non-commutativity parameter is removed. The Grisaru-Penati version [3, 4] introduces a constraint which is non-trivial only in the non-commutative case. The constraint is required by integrability but it is satisfied by the one-soliton solutions. However, at the quantum level this model gives rise to particle production as was discovered by evaluating tree-level scattering amplitudes [4]. On the other hand, introducing an auxiliary field, Lechtenfeld et al. [6] proposed a novel NCSG model which seems to possess a factorizable and causal S-matrix.

Recently, in ordinary commutative space the so-called $sl(2)$ affine Toda model coupled to matter (Dirac) fields (ATM) has been shown to be a Master lagrangian (ML) from which one can derive the sine-Gordon and massive Thirring models, describing the strong/weak phases of the model, respectively [7]–[10]. Besides, the ML approach was successfully applied in the non-commutative case to uncover related problems in $(2+1)$ dimensions regarding the duality equivalence between the Maxwell-Chern-Simons theory (MCS) and the Self-Dual (SD) model [11].

In this paper we extend some properties of the ordinary $sl(2)$ ATM model to the NC case. We show that replacing the products of fields by the $\star-$products, on the level of its effective action, the ATM theory is still an integrable field theory. Since the ordinary effective action gives rise to equations of motion which can be derived from a zero-curvature equation, we may alternatively construct the NC extension of the ATM model directly starting from its zero-curvature formulation. In this way the ATM model belongs to those class of integrable field theories in which the direct replacement of the $\star-$product in the action turns out the model still integrable [5]. However, in our case the NC extension of the WZW term in the ATM effective action must be considered properly.

The study of these models become interesting since the $su(n)$ ATM theories constitute excellent laboratories to test ideas about confinement [10, 12], the role of solitons in quantum field theories [7], duality transformations interchanging solitons and particles [7, 13], as well as the reduction processes of the (two-loop) Wess-Zumino-Novikov-Witten (WZNW) theory from which the ATM models are derivable [14, 9]. Moreover, the ATM type systems may also describe some low dimensional condensed matter phenomena, such as self-trapping of electrons into solitons, see e.g. [15], tunnelling in the integer quantum Hall effect [16], and, in particular, polyacetylene molecule systems in connection with fermion number fractionization [17]. It has been shown that the $su(2)$ ATM model describes the low-energy spectrum of QCD_2 (*one flavor* and N colors in the fundamental and $N=2$ in the adjoint representations, respectively) [12].

The paper is organized as follows. In the next section we present the NC extensions of the ATM model relevant to our discussions. This procedure deals with the choice of the group for the Toda field g. We introduce two types of master lagrangians ($\mathrm{NCATM}_{1,2}$), the *first* one defined for $g \in \mathrm{U}(1)\times\mathrm{U}(1)$ with the same content of matter fields as the ordinary ATM; the *second* one defined for two copies of the usual ATM such that $g,\, \bar{g} \in \mathrm{U}(1)_C$. In section 3 the $\mathrm{NCSG}_{1,2}$ models are derived from the relevant master lagrangians through reduction procedures resembling the one performed in the ordinary ATM $\rightarrow$ SG reduction. In section 4 we decouple on shell the theories $\mathrm{NCSG}_{1,2}$ and $\mathrm{NCMT}_{1,2}$, respectively. This procedure is further justified in sections 5 and 6 when we consider the $\mathrm{NCMT}_{1,2}$ lagrangians and the soliton mappings satisfying the relevant decoupling equations. In section 5 we consider the (bosonic) $\mathrm{NCMT}_{1,2}$ models, as well as their global symmetries, associ-

JHEP03(2005)037

ated currents and integrability properties. In these developments the double-gauging of a U(1) symmetry in the star-localized Noether procedure to get the currents deserve a careful treatment. In section 6 we present the soliton solutions and establish the strong-weak duality between the $\mathrm{NCSG}_{1,2}$ and $\mathrm{NCMT}_{1,2}$ (real soliton sector of models 2) models. The section 7 presents the conclusions and possible future directions. The appendix A provides the affine $sl(2)$ Lie algebra properties. Some results of the ordinary ATM model are summarized in appendix B.

2. The NC affine Toda coupled to matter $(\mathrm{NCATM})_{1,2}$

The commutative Toda field g in (B.4) belongs to the complexified $\mathrm{U}(1)_C$ group since in general $\varphi \in \mathbb{C}$. Different NC extensions of the ATM model (B.15) are possible as long as all of them reproduce in the commutative limit the equations of motion (B.6)–(B.8). The symmetry group of the ordinary SL(2) ATM model (see appendices A and B) in the NC case is not closed under $\star$; then, the NC extension requires the GL(2) group. In the next steps we define two versions of the non-commutative GL(2) affine Toda model coupled to matter ($\mathrm{NCATM}_{1,2}$). Let us define the *first* NC extension (NCATM_1) as

$$
\begin{aligned}
S_{NCATM_1} &\equiv S\left[g, W^{\pm}, F^{\pm}\right] \\
&= I_{\mathrm{WZW}}[g] + \int d^2x \Big\{ \frac{1}{2}\left\langle \partial_- W^- \star [E_2, W^-] \right\rangle - \frac{1}{2}\left\langle [E_{-2}, W^+] \star \partial_+ W^+ \right\rangle + \\
&\qquad + \left\langle F^- \star \partial_+ W^+ \right\rangle + \left\langle \partial_- W^- \star F^+ \right\rangle + \\
&\qquad + \left\langle F^- \star g \star F^+ \star g^{-1} \right\rangle \Big\},
\end{aligned}
\tag{2.1}
$$

where $F \star G = F \exp\left(\frac{\theta}{2}(\overleftarrow{\partial_+}\overrightarrow{\partial_-} - \overleftarrow{\partial_-}\overrightarrow{\partial_+}) \right) G$, and $g \in \mathrm{U}(1) \times \mathrm{U}(1)$. $I_{\mathrm{WZW}}[g]$ is the NC generalization of a WZNW action for g

$$
I_{\mathrm{WZW}}[g] = \int d^2x \left[\partial_+ g \star \partial_- g^{-1} + \int_0^1 dy \hat{g}^{-1} \star \partial_y \hat{g} \star \left[\hat{g}^{-1} \star \partial_+ \hat{g}, \hat{g}^{-1} \star \partial_- \hat{g} \right]_\star \right], \tag{2.2}
$$

where the homotopy path $\hat{g}(y)$ such that $\hat{g}(0) = 1$, $\hat{g}(1) = g$ ($[y, x_+] = [y, x_-] = 0$) has been defined. The WZW term in this case gives a non-vanishing contribution due to the noncommutativity. This is in contrast with the action (B.15) in ordinary space. Notice that we have introduced two independent fields (one real field for each U(1) group) instead of the complex field φ. This is justified since in the NC realm the abelian subgroup of GL(2) fails to decouple from the rest of the fields of the model as we will show below.

From (2.1) one can derive the set of equations of motion for the corresponding fields

$$
\partial_-(g^{-1} \star \partial_+ g) = \left[F^-, \qquad g \star F^+ \star g^{-1} \right]_\star \tag{2.3}
$$

$$
\partial_+ F^- = [E_{-2}, \partial_+ W^+], \qquad \partial_- F^+ = -[E_2, \partial_- W^-], \tag{2.4}
$$

$$
\partial_+ W^+ = -g \star F^+ \star g^{-1}, \qquad \partial_- W^- = -g^{-1} \star F^- \star g. \tag{2.5}
$$

JHEP03(2005)037

Substituting the derivatives of $W^{\pm}$'s given in the eqs. (2.5) into the eqs. (2.4) one can get the equivalent set of equations

$$\partial_+ F^- = -[E_{-2}\,,\, g\star F^+ \star g^{-1}]\,, \qquad \partial_- F^+ = [E_2\,,\, g^{-1}\star F^- \star g]\,. \tag{2.6}$$

Notice that in the action (2.1) one can use simultaneously the cyclic properties of the group trace and the $\star$ product. Then, the action (2.1) and the equations of motion (2.3)-(2.5) have the left-right local symmetries given by

$$g \to h_L(x_-)\star g(x_+,x_-)\star h_R(x_+)\,, \tag{2.7}$$

$$F^+ \to h_R^{-1}(x_+)\star F^+(x_+,x_-)\star h_R(x_+)\,, \quad W^- \to h_R^{-1}(x_+)\star W^-(x_+,x_-)\star h_R(x_+)\,, \tag{2.8}$$

$$F^- \to h_L(x_-)\star F^-(x_+\,,x_-)\star h_L^{-1}(x_-)\,, \quad W^+ \to h_L(x_-)\star W^+(x_+,x_-)\star h_L^{-1}(x_-)\,. \tag{2.9}$$

In fact, the system of eqs. (2.3)–(2.5) is invariant under the above symmetries if the following conditions are supplied

$$h_R(x_+)\star E_2\, h_R^{-1}(x_+) \;=\; E_2\,, \qquad h_L^{-1}(x_-)\star E_{-2}\, h_L(x_-) \;=\; E_{-2}\,, \tag{2.10}$$

where $h_{L/R}(x_{\mp}) \in \mathcal{H}_0^{L/R}$, $\mathcal{H}_0^{L/R}$ being abelian sub-groups of $\mathrm{GL}(2,\mathbb{C})$.

Next, we define the *second* version of the NC affine Toda model coupled to matter NCATM_2 as

$$S_{NCATM_2} \equiv S[g,W^{\pm},F^{\pm}] + S[\bar{g},\mathcal{W}^{\pm},\mathcal{F}^{\pm}]\,, \tag{2.11}$$

where the independent fields g and $\bar{g}$, related to the set of matter fields $\{W^{\pm},F^{\pm}\}$ and $\{\mathcal{W}^{\pm},\mathcal{F}^{\pm}\}$, respectively, belong to complexified $\mathrm{U}(1)_C$ groups with the action $S[.\,,\,.\,,\,.]$ being a Moyal extension of (B.15) for $g \in \mathrm{U}(1)_C$.

The equations of motion for the NCATM_2 model (2.11) comprise the eqs. (2.3)–(2.5) written for $g \in \mathrm{U}(1)_C$ and a set of analogous equations for the remaining fields $\bar{g}$, $\mathcal{F}^{\pm}$ and $\mathcal{W}^{\pm}$. Moreover, in addition to the symmetry transformations (2.7)–(2.9) one must consider similar expressions for $\bar{g}$, $\mathcal{F}^{\pm}$ and $\mathcal{W}^{\pm}$.

3. NC versions of the sine-Gordon model ($\mathrm{NCSG}_{1,2}$)

In order to derive the NC versions of the sine-Gordon model we follow the master lagrangian approach [18], as in the ordinary SG derivation [9], starting from the $\mathrm{NCATM}_{1,2}$ models (2.1) and (2.11), respectively. Let us concentrate first on the equations of motion (2.3)–(2.5) which are understood to be written for $g \in \mathrm{U}(1)\times\mathrm{U}(1)$ or $\mathrm{U}(1)_C$. We proceed by considering the eqs. (2.4) and integrating them

$$F^- = [E_{-2},W^+] + f^-(x_-)\,, \qquad F^+ = -[E_2,W^-] - f^+(x_+)\,, \tag{3.1}$$

with the $f^{\pm}(x_{\pm})$'s being analytic functions. Next, we replace the $F^{\pm}$ of eqs. (3.1) and the $\partial_{\pm}W^{\pm}$ of (2.5), written in terms of $W^{\pm}$, into the action (2.1) to get

$$\begin{aligned} S'[g,W^{\pm},f^{\pm}] = I_{\mathrm{WZW}}[g] + \int d^2x \Big\{ &\frac{1}{2}\left\langle [E_{-2},W^+]\star g\star f^+\star g^{-1}\right\rangle + \\ &+\frac{1}{2}\left\langle g^{-1}\star f^-\star g\star[E_2,W^-]\right\rangle + \\ &+\left\langle g^{-1}\star f^-\star g\star f^+\right\rangle \Big\}\,. \end{aligned} \tag{3.2}$$

As the next step, one writes the equations of motion for the $f^{\pm}(x_{\pm})$'s and solves for them; afterwards, substitutes those expressions into the intermediate action (3.2) getting

$$S''[g, W^{\pm}] = I_{\rm WZW}[g] - \frac{1}{4}\int d^2x \left[\langle [E_{-2}, W^{+}] \star g \star [E_2, W^{-}] \star g^{-1}\rangle\right]. \qquad (3.3)$$

Notice that (3.3) has inherited from the NCATM action the local symmetries (2.7)–(2.9). Therefore, one considers the gauge fixing

$$2i\Lambda^{-} = [E_{-2}, W^{+}], \qquad 2i\Lambda^{+} = [E_2, W^{-}], \qquad (3.4)$$

where $\Lambda^{\pm} \in \hat{\mathcal{G}}_{\pm 1}$ are some constant generators in the subsets of grade ± 1 (A.9).

Then for this gauge fixing the effective action (3.3) becomes

$$\begin{aligned} S_{NCSG_1}[g] &\equiv S[g] \\ &= I_{\rm WZW}[g] + \int d^2x \left[\langle \Lambda^{-} \star g \star \Lambda^{+} \star g^{-1}\rangle\right]. \end{aligned} \qquad (3.5)$$

Then, the equation of motion for the field g is

$$\partial_{-}(g^{-1} \star \partial_{+} g) = \Big[\Lambda^{-}\,,\, g \star \Lambda^{+} g^{-1}\Big]. \qquad (3.6)$$

The action (3.5) for $g \in \mathrm{U}(1) \times \mathrm{U}(1)$ defines the *first* version of the non-commutative sine-Gordon model (NCSG_1) (see section 3.1).

Since the above reduction process can be carried over verbatim for each sector of the NCATM_2 model (2.11) and its independent set of fields $\{g, F^{\pm}, W^{\pm}\}$ and $\{\bar{g}, \mathcal{F}^{\pm}, \mathcal{W}^{\pm}\}$, one can write in this case as the reduced model

$$S_{NCSG_2}[g, \bar{g}] = S[g] + S[\bar{g}]\,; \qquad g,\, \bar{g} \in \mathrm{U}(1)_C. \qquad (3.7)$$

The equations of motion derived from this action become the eq. (3.6) written for $g \in \mathrm{U}(1)_C$ and

$$\partial_{-}(\bar{g}^{-1} \star \partial_{+} \bar{g}) = \Big[\Lambda^{-}\,,\, \bar{g} \star \Lambda^{+} \bar{g}^{-1}\Big]. \qquad (3.8)$$

The action (3.7) defines the *second* version of the non-commutative sine-Gordon model (NCSG_2) (see section 3.2).

In the subsections below we will see that NCSG_1 (3.5) written for $g \in \mathrm{U}(1) \times \mathrm{U}(1)$ and NCSG_2 (3.7) are precisely the Lechtenfeld et al. [6] and Grisaru-Penati [4] proposals for the NC versions of the SG model, respectively.

3.1 Lechtenfeld et al. proposal (NCSG_1)

The NCSG_1 version has been obtained through the reduction process starting from the NCATM_1 model (2.1), so let us write the field $g \in \mathrm{U}(1) \times \mathrm{U}(1)$ in the representation

$$g = \begin{pmatrix} e_{\star}^{i\varphi_{+}} & 0 \\ 0 & e_{\star}^{-i\varphi_{-}} \end{pmatrix} \equiv g_{+} g_{-}\,, \quad g_{+} = \begin{pmatrix} e_{\star}^{i\varphi_{+}} & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{-} = \begin{pmatrix} 1 & 0 \\ 0 & e_{\star}^{-i\varphi_{-}} \end{pmatrix} \qquad (3.9)$$

with $\varphi_{\pm}$ being real fields.

For the Λ's taken as

$$\Lambda^+ = M(E_+^0 + E_-^1)\,, \qquad \Lambda^- = M(E_-^0 + E_+^{-1})\,, \tag{3.10}$$

the action (3.5) for g given in (3.9), upon using the Polyakov-Wiegmann identity, can be written as[1]

$$S_{NCSG_1}[g_+, g_-] = I_{\rm WZW}[g_+] + I_{\rm WZW}[g_-] + M^2 \int d^2x \left(e_\star^{i\varphi_+} \star e_\star^{i\varphi_-} + e_\star^{-i\varphi_-} \star e_\star^{-i\varphi_+} - 2 \right). \tag{3.11}$$

In this way we have re-derived the Lechtenfeld et al. action (NCSG_1) for the NC sine-Gordon [6]. The Eqs. of motion become

$$\partial_- \left(e_\star^{-i\varphi_+} \star \partial_+ e_\star^{i\varphi_+} \right) = -M^2 \left(e_\star^{i\varphi_+} \star e_\star^{i\varphi_-} - e_\star^{-i\varphi_-} \star e_\star^{-i\varphi_+} \right); \tag{3.12}$$

$$\partial_- \left(e_\star^{i\varphi_-} \star \partial_+ e_\star^{-i\varphi_-} \right) = +M^2 \left(e_\star^{i\varphi_+} \star e_\star^{i\varphi_-} - e_\star^{-i\varphi_-} \star e_\star^{-i\varphi_+} \right). \tag{3.13}$$

In the $\theta \to 0$ limit the above equations can be written as

$$\partial_- \partial_+ \left(\varphi_+ - \varphi_- \right) = 0\,; \tag{3.14}$$

$$\partial_- \partial_+ \left(\varphi_+ + \varphi_- \right) = -4M^2 \sin\left(\varphi_+ + \varphi_- \right). \tag{3.15}$$

If we choose $\varphi_+ = \varphi_- \equiv \frac{1}{2}\varphi_{SG} \leftrightarrow e^{i\varphi_+} = e^{i\varphi_-} \in \mathrm{U}(1)_A$, we have in (3.15) the SG equation $\partial^2 \varphi_{SG} = -4M^2 \sin(\varphi_{SG})$. Thus the $\mathrm{U}(1)_V$ degree of freedom completely decouples in the commutative limit. Then, in the model NCSG_1 one can define the topological charge as

$$Q_{\rm topol.}^{NCSG_1} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \frac{d\left(\varphi_+ + \varphi_- \right)}{dx} \equiv \sum_n \theta^n Q_{NCSG_1}^{(n)}\,. \tag{3.16}$$

The Leznov-Saveliev formulation of the NCSG_1 model through a zero-curvature equation will lead to (3.6) for the parametrization (3.9). A linear system for this system is provided in [6] through a dimensional reduction from $(2+2)$ self-dual Yang Mills theory. Then, following a similar procedure to that developed in [5] (see also [19]) for a NC linear system one can construct infinite conserved currents.

3.2 The Grisaru-Penati proposal (NCSG_2)

The second NC version of the sine-Gordon system (NCSG_2) is written in terms of the following representation of the complexified $\mathrm{U}(1)_C$ group elements

$$g = e_\star^{i\varphi H^0} \equiv \begin{pmatrix} e_\star^{i\varphi} & 0 \\ 0 & e_\star^{-i\varphi} \end{pmatrix} \qquad \text{and} \qquad \bar{g} = e_\star^{i(\varphi)^\dagger H^0} \equiv \begin{pmatrix} e_\star^{i\varphi^\dagger} & 0 \\ 0 & e_\star^{-i\varphi^\dagger} \end{pmatrix}, \tag{3.17}$$

where the field φ is a general complex field and the g and $\bar{g}$ are formally considered as independent fields.

[1] Assuming the general forms $\Lambda^+ = (\Lambda_R\, E_+^0 + \widetilde{\Lambda}_R E_-^1)$, $\Lambda^- = (\Lambda_L\, E_+^{-1} + \widetilde{\Lambda}_L\, E_-^0)$, one gets $S[g] = I_{\rm WZW}[g] + \int [\Lambda_L \widetilde{\Lambda}_R e_\star^{-i\varphi_-} \star e_\star^{-i\varphi_+} + \widetilde{\Lambda}_L \Lambda_R e_\star^{i\varphi_+} \star e_\star^{i\varphi_-}]$, which upon setting $\widetilde{\Lambda}_L \Lambda_R = 2e^{i\delta} M^2$, $(\delta = 0)$ reproduces (3.11) (the phase $\delta \neq 0$ can be absorbed by shifting the fields $\varphi_\pm$).

The master lagrangian from which the NCSG_2 model originates is the NCATM_2 theory (2.11). Thus, one must consider the reduced model (3.7).

Thus for the Λ's given in (3.10) the action (3.7), taking into account the action (3.5) written for $g \in \mathrm{U}(1)_C$, can be written as

$$S_{NCSG_2} = I_{\mathrm{WZW}}[g] + M^2 \int d^2x \,\mathrm{Tr}\Big(g^2 + g^{-2} - 2\Big) + \\ + I_{\mathrm{WZW}}[\bar{g}] + M^2 \int d^2x \,\mathrm{Tr}\Big(\bar{g}^2 + \bar{g}^{-2} - 2\Big) \tag{3.18}$$

where $I_{\mathrm{WZW}}[g]$ is the NC generalization of a complexified U(1) WZNW action [20].

In this way we have arrived at the Grisaru-Penati proposal for the NC sine-Gordon system (NCSG_2) [3, 4].

Notice that, when the field φ is real, one has $g = \bar{g}$ and the action reduces to $S_{NCSG_2}[g, \bar{g}] = 2[I_{\mathrm{WZW}}[g] + \int M^2\,\mathrm{Tr}(g^2 + g^{-2} - 2)]$. In fact, it is possible to find real solutions for the NCSG_2 model [3].

Regarding the $\mathrm{NCSG}_{1,2}$ relationships, notice that assuming general complex fields $\varphi_\pm$ and imposing the reduction $\varphi_+ = \varphi_- \equiv \varphi$ in the NCSG_1 action (3.11) one may get the $S[g]$ sector of the NCSG_2 model (3.18).

The Leznov-Saveliev formulation of the NCSG_2 model [5] through a zero-curvature equation leads to eqs. (3.6) and (3.8).

4. Decoupling of $\mathrm{NCSG}_{1,2}$ and $\mathrm{NCMT}_{1,2}$ models

In the study of the ordinary (commutative) ATM model performed in [7]–[9], the massive Thirring model (MT) was obtained by means of a hamiltonian reduction and the so-called decoupling procedures. The first procedure requires the definition of conjugated momenta for the fields of the model. In the NC case this procedure encounters some complications due to the infinite sum of time derivatives implicit in the Moyal product (see, e.g. [21]) and then we must resort to an alternative method to uncover the MT sector of the NCATM theories. In [7, 9] it has been proposed another approach to recover the SG and MT models out of the ordinary ATM model. This proceeds by decoupling the set of equations of the ATM model into the corresponding dual models. This procedure can be adopted in the NC case by writing a set of mappings between the fields of the model such that the eqs. (2.3) and (2.6) when rewritten using those mappings completely decouple the scalar and the matter fields. Following the commutative case let us consider the mappings

$$\Big[F^-\,,\, g\star F^+ \star g^{-1}\Big]_\star = \Big[\Lambda^-\,,\, g\star \Lambda^+ \star g^{-1}\Big] \tag{4.1}$$

$$-\Big[E_{-2}, g\star F^+ \star g^{-1}\Big] = \Big[E_{-2}, \Big[E_2, W^-\Big]\Big] - \frac{\lambda}{8}\Big[\Big[E_{-2}, W^+\Big], \Big[\Big[E_2, W^-\Big], W^-\Big]_\star\Big]_\star \tag{4.2}$$

$$\Big[E_2, g^{-1}\star F^- \star g\Big] = \Big[E_2, \Big[E_{-2}, W^+\Big]\Big] - \frac{\lambda}{8}\Big[\Big[E_2, W^-\Big], \Big[\Big[E_{-2}, W^+\Big], W^+\Big]_\star\Big]_\star \tag{4.3}$$

$$F^\pm = \mp[E_{\pm 2}\,,\, W^\mp]\,. \tag{4.4}$$

In the relations above the field g comes from section 2 and we assume it belongs to either $\mathrm{U}(1)\times\mathrm{U}(1)$ or $\mathrm{U}(1)_C$. In writing the mappings (4.1)–(4.4) a helpful organizing guide is the principal gradation structure (see appendix A) such that only equal grade terms (0 or ±1) appear in each relationship.

It is clear that the $\mathrm{NCSG}_{1,2}$ (one sector of model 2) equation of motion (3.6) is recovered from the equation of motion (2.3) and the decoupling equation (4.1). We expect that a noncommutative version of the massive Thirring model (NCMT_1) defined for the fields $W^{\pm}$, corresponding to the Letchenfeld et al. version NCSG_1, will emerge from the decoupling eqs. (4.2)–(4.4) and the eqs. of motion (2.6).

In order to recover the Grisaru-Penati version NCSG_2 one must write similar decoupling expressions for the full set of fields $\{g, F^{\pm}, W^{\pm}\}$ and $\{\bar{g}, \mathcal{F}^{\pm}, \mathcal{W}^{\pm}\}$. Thus, following similar steps to the previous construction we expect to recover another version of the NC massive Thirring model NCMT_2 defined for the fields $\{W^{\pm}, \mathcal{W}^{\pm}\}$. In the next section we propose two versions of the non-commutative massive Thirring theories ($\mathrm{NCMT}_{1,2}$) by providing the lagrangians and the zero-curvature equations.

5. The NC (Bosonic) Thirring models (B)$\mathrm{NCMT}_{1,2}$

In ordinary space the formulation of the MT model can be performed in two ways. First, the classical fields can be assumed to be anti-commuting grassmannian fields [22]. Second, the fields $\psi, \widetilde{\psi}$ considered as ordinary commuting fields define the so-called bosonic massive Thirring (BMT) [23, 24]. Even though in [8, 9] the authors have been considered anti-commuting fields in order to make the reduction procedure of the relevant ATM models into its dual theories, here, we follow the second formulation, i.e. we will consider commuting fields. The reason is that the zero-curvature formulations of the $\mathrm{NCMT}_{1,2}$ models follow from that of the ATM relevant formulation in the context of the affine Lie algebra $sl(2)$ construction. This point of view is also in accordance with the assumption in [14] where these fields have been considered as ordinary commuting fields leaving the discussion of their fermionic character to the full quantum treatment of the models, since the statistics of fields in two-dimensions depends upon the coupling constant. Regarding this point, it is known that already in the (B)MT classical solutions (see, e.g. [23]) it has been discussed the appearance of certain Pauli exclusion principle associated with the multi-soliton solutions in the context of the classical correspondence between the SG and the MT models.

The decoupling procedure of the $\mathrm{NCAMT}_{1,2}$ models provide two models which we shall call (bosonic) non-commutative massive Thirring models $[\mathrm{(B)NCMT}]_{1,2}$, respectively, in the derivations below.

5.1 (B)NCMT_1

We propose the (B)NCMT_1 lagrangian related to the fields $W^{\pm}$ such that it reproduces the relevant equations of motion we have outlined in the last section by the decoupling procedure of the NCATM model. Let us consider the action

$$S[W^{\pm},\widetilde{W}^{\pm}] = \int\Big\{\Big\langle[E_{-2},\widetilde{W}^{+}]\star\partial_{+}W^{+}\Big\rangle - \Big\langle\partial_{-}W^{-}\star[E_{2},\widetilde{W}^{-}]\Big\rangle - \\ - \Big\langle[E_{-2},\widetilde{W}^{+}]\star[E_{2},W^{-}]\Big\rangle - \Big\langle[E_{-2},W^{+}]\star[E_{2},\widetilde{W}^{-}]\Big\rangle - \\ - \lambda\left\langle J^{-}\star J^{+}\right\rangle\Big\}, \tag{5.1}$$

where the current components are given by

$$J^{+} = \frac{1}{4}\Big([[E_{-2},\widetilde{W}^{+}],W^{+}]_{\star} + [[E_{-2},W^{+}],\widetilde{W}^{+}]_{\star}\Big) \tag{5.2}$$

$$J^{-} = -\frac{1}{4}\Big([[E_{2},W^{-}],\widetilde{W}^{-}]_{\star} + [[E_{2},\widetilde{W}^{-}],W^{-}]_{\star}\Big). \tag{5.3}$$

In order to write the expressions in more symmetric form we have considered additional fields denoted by $\widetilde{W}^{\pm}$ (see below). We will show that the action (5.1) is related to the NCMT_1 version. In the derivations below we use the explicit matrix representation (A.1) for the GL(2) generators and its corresponding loop extension. The field components are defined by

$$W^{+} = \sqrt{\frac{4i}{m_{\psi}}}\Big(\psi_L E^{0}_{+} + \widetilde{\psi}_L E^{1}_{-}\Big)\,,\; W^{-} = -\sqrt{\frac{4i}{m_{\psi}}}\Big(\psi_R E^{-1}_{+} - \widetilde{\psi}_R E^{0}_{-}\Big) \tag{5.4}$$

$$\Big[E_{-2}\,,\,\widetilde{W}^{+}\Big] = -\sqrt{\frac{im_{\psi}}{4}}\Big(\psi_L E^{-1}_{+} - \widetilde{\psi}_L E^{0}_{-}\Big)\,,\;\Big[E_{2}\,,\,\widetilde{W}^{-}\Big] = \sqrt{\frac{im_{\psi}}{4}}\Big(\psi_R E^{0}_{+} + \widetilde{\psi}_R E^{1}_{-}\Big). \tag{5.5}$$

Then the $\mathrm{(B)NCMT}_1$ action (5.1) in terms of the field components is given by

$$\mathcal{S}_{(B)NCMT_1} = \int d^2x\Big[2i\widetilde{\psi}_L\partial_{+}\psi_L + 2i\widetilde{\psi}_R\partial_{-}\psi_R - im_{\psi}\Big(\widetilde{\psi}_R\psi_L - \widetilde{\psi}_L\psi_R\Big) - \\ - \lambda\Big(\widetilde{\psi}_R\star\psi_R\star\widetilde{\psi}_L\star\psi_L + \psi_R\star\widetilde{\psi}_R\star\psi_L\star\widetilde{\psi}_L\Big)\Big]. \tag{5.6}$$

Notice that although $W^{\pm}$ and $\widetilde{W}^{\pm}$ are proportional, we consider them as independent fields since the waved fields appear inside the Lie bracket making the expressions $[E_{\pm2}\,,\,\widetilde{W}^{\mp}]$ indeed independent from the fields $W^{\pm}$. This notation will be useful in order to derive the field equations in matrix form starting from the action (5.1).

By taking the functional derivative of (5.1) with respect to $\widetilde{W}^{+}$, $\widetilde{W}^{-}$, W^{+} and W^{-}, respectively, one can get the equations of motion

$$\Big[E_{-2}\,,\,\partial_{+}W^{+}\Big] = \Big[E_{-2},\Big[E_{2},W^{-}\Big]\Big] + \frac{\lambda}{8}\Big[\Big[E_{-2},W^{+}\Big],\Big[\Big[E_{2},\widetilde{W}^{-}\Big],W^{-}\Big]_{\star} + \\ + \Big[\Big[E_{2},W^{-}\Big],\widetilde{W}^{-}\Big]_{\star}\Big]_{\star} \tag{5.7}$$

$$\left[E_2\,,\,\partial_- W^-\right] = -\left[E_2,\left[E_{-2},W^+\right]\right] - \frac{\lambda}{8}\left[\left[E_2,W^-\right],\left[\left[E_{-2},\widetilde{W}^+\right],W^+\right]_\star + \right.$$
$$\left. + \left[\left[E_{-2},W^+\right],\widetilde{W}^+\right]_\star\right]_\star . \tag{5.8}$$
$$\left[E_{-2}\,,\,\partial_+ \widetilde{W}^+\right] = \left[E_{-2},\left[E_2,\widetilde{W}^-\right]\right] + \frac{\lambda}{8}\left[\left[E_{-2},\widetilde{W}^+\right],\left[\left[E_2,W^-\right],\widetilde{W}^-\right]_\star + \right.$$
$$\left. + \left[\left[E_2,\widetilde{W}^-\right],W^-\right]_\star\right]_\star \tag{5.9}$$
$$\left[E_2\,,\,\partial_- \widetilde{W}^-\right] = -\left[E_2,\left[E_{-2},\widetilde{W}^+\right]\right] - \frac{\lambda}{8}\left[\left[E_2,\widetilde{W}^-\right],\left[\left[E_{-2},W^+\right],\widetilde{W}^+\right]_\star + \right.$$
$$\left. + \left[\left[E_{-2},\widetilde{W}^+\right],W^+\right]_\star\right]_\star . \tag{5.10}$$

The eqs. (5.9)–(5.10) when considered in terms of the field components (5.4)–(5.5) are corresponding copies of the eqs. (5.7)–(5.8), so in the considerations below it will be sufficient to pay attention only on these equations. One can verify that the equations of motion (5.7)–(5.8) reproduce the set of equations obtained when the decoupling mappings (4.2)–(4.4) are replaced into the equations (2.6).

The equations of motion (5.7)–(5.8) for the fields defined by (5.4)–(5.5) become

$$\partial_+\psi_L = -\frac{m_\psi}{2}\psi_R - i\,\frac{\lambda}{2}\left(\psi_L\star\widetilde{\psi}_R\star\psi_R + \psi_R\star\widetilde{\psi}_R\star\psi_L\right), \tag{5.11}$$
$$\partial_-\psi_R = \frac{m_\psi}{2}\psi_L - i\,\frac{\lambda}{2}\left(\psi_R\star\widetilde{\psi}_L\star\psi_L + \psi_L\star\widetilde{\psi}_L\star\psi_R\right) \tag{5.12}$$
$$\partial_+\widetilde{\psi}_L = -\frac{m_\psi}{2}\widetilde{\psi}_R + i\,\frac{\lambda}{2}\left(\widetilde{\psi}_L\star\psi_R\star\widetilde{\psi}_R + \widetilde{\psi}_R\star\psi_R\star\widetilde{\psi}_L\right), \tag{5.13}$$
$$\partial_-\widetilde{\psi}_R = \frac{m_\psi}{2}\widetilde{\psi}_L + i\,\frac{\lambda}{2}\left(\widetilde{\psi}_R\star\psi_L\star\widetilde{\psi}_L + \widetilde{\psi}_L\star\psi_L\star\widetilde{\psi}_R\right). \tag{5.14}$$

Notice that in the limit $\theta \to 0$ the equations (5.11)–(5.14) reduce to the usual (B)MT Eqs. of motion for $\widetilde{\psi}_{R,\,L} = \psi^\star_{R,\,L}$ ($\star$ here means complex conjugation) [23, 24].

The system of eqs. (5.7)–(5.8) admit a zero-curvature formulation. In fact, consider

$$A_- = E_{-2} + i\sqrt{\frac{\lambda}{4}}\,\left[E_{-2}\,,\,W^+\right] + \frac{\lambda}{4}\left[[E_{-2}\,,\,\widetilde{W}^+],\,W^+\right]_\star\,, \tag{5.15}$$
$$A_+ = -E_2 - i\sqrt{\frac{\lambda}{4}}\,\left[E_2\,,\,W^-\right] - \frac{\lambda}{4}\left[[E_2\,,\,\widetilde{W}^-],\,W^-\right]_\star\,. \tag{5.16}$$

Then from the zero-curvature condition $[\partial_+ + A_+\,,\,\partial_- + A_-\,]_\star = 0$ one obtains the set of eqs. (5.7)–(5.8) plus an additional equation

$$\partial_+[[E_{-2}\,,\,\widetilde{W}^+]\,,\,W^+]_\star + \partial_-[[E_2\,,\,\widetilde{W}^-]\,,\,W^-]_\star =$$
$$= \left[[E_2\,,\,W^-]\,,\,[E_{-2}\,,\,W^+]\right] - \frac{\lambda}{4}\left[[[E_2\,,\,\widetilde{W}^-]\,,\,W^-]_\star\,,\,[[E_{-2}\,,\,\widetilde{W}^+]\,,\,W^+]_\star\right]. \tag{5.17}$$

The eq. (5.17) in terms of the component fields gives rise to the equation

$$\partial_-(\widetilde{\psi}_R\star\psi_R)-\partial_+(\widetilde{\psi}_L\star\psi_L)=m_\psi(\widetilde{\psi}_R\star\psi_L+\widetilde{\psi}_L\star\psi_R)- \\ -i\,\lambda(\widetilde{\psi}_R\star\psi_R\star\widetilde{\psi}_L\star\psi_L-\widetilde{\psi}_L\star\psi_L\star\widetilde{\psi}_R\star\psi_R) \quad (5.18)$$

and another equation obtained from (5.18) by conveniently substituting

$$\{\psi_R,\,\psi_L\}\leftrightarrow\{\widetilde{\psi}_R,\,\widetilde{\psi}_L\}\qquad\text{and}\qquad i\to -i\,. \quad (5.19)$$

The set of eqs. (5.18) and the one obtained by (5.19) may be shown to be satisfied as the result of the field equations (5.11)–(5.14). The eq. (5.18) can be written as $\partial_\mu j_5^{(1)\,\mu}=2im_\psi\bar{\psi}\gamma_5\star\psi-2i\,\lambda(\widetilde{\psi}_R\star\psi_R\star\widetilde{\psi}_L\star\psi_L-\widetilde{\psi}_L\star\psi_L\star\widetilde{\psi}_R\star\psi_R)$, where $j_5^{(1)\,\mu}\equiv\bar{\psi}\gamma^\mu\gamma_5\star\psi$. In the $\theta\to 0$ limit the last term inside parenthesis in (5.18) vanishes and the $j_5^{(1)\,\mu}$ current is conserved for $m_\psi=0$. The expression for $\partial_\mu j_5^{(2)\,\mu}$ may be obtained by making the substitutions (5.19) in the relevant terms of $\partial_\mu j_5^{(1)\,\mu}$.

The currents (5.2)–(5.3) satisfy $\partial_+J^+ + \partial_-J^- = 0$ or equivalently written in field components

$$\partial_-\left(\widetilde{\psi}_R\star\psi_R\right)+\partial_+\left(\widetilde{\psi}_L\star\psi_L\right)=0\,;\qquad \partial_-\left(\psi_R\star\widetilde{\psi}_R\right)+\partial_+\left(\psi_L\star\widetilde{\psi}_L\right)=0\,. \quad (5.20)$$

The (B)NCMT$_1$ model in (5.1) [or (5.6)] has a global U(1) symmetry. In order to obtain the currents by the Noether procedure we make the global transformation localized, as discussed in [25] this is not unique in the NC case. In the equations (5.20) one recognizes the currents associated to U(1) $\times$ U(1) symmetry implemented in NC space with the transformation rules

$$\psi\to U_2(x)\star\psi\star \mathrm{U}_1^{-1}(x)\,;\qquad \widetilde{\psi}\to \mathrm{U}_1(x)\star\widetilde{\psi}\star\mathrm{U}_2^{-1}(x)\,,\qquad \mathrm{U}_{1,2}(x)=e_\star^{i\alpha_{1,2}}\,, \quad (5.21)$$

where $\mathrm{U}_{1,2}(x)$ are independent starred exponentials with $\alpha_{1,2}=$ real functions. In fact, the eqs. (5.21) are the most general transformations in NC space for a charged field [25].

The $\mathrm{U}_1(1)$ global symmetry of the action (5.6) gives through the Noether procedure the conservation equation $\partial_\mu j^{(1)\,\mu}=0$; $j^{(1)\,\mu}\equiv\bar{\psi}\star\gamma^\mu\psi$, where $\bar{\psi}=\widetilde{\psi}^T\gamma^0$, corresponding to the first Eq. in (5.20). The another $U_2(1)$ current conservation equation becomes $\partial_\mu j^{(2)\,\mu}=0; j^{(2)\,\mu}\equiv -\psi^T\gamma^0\star\gamma^\mu\widetilde{\psi}$ and corresponds to the second eq. in (5.20). Since the charge is associated to global transformation of the charged field for which there is no difference between the ordinary and non-commutative product one can conclude that the currents share the same charge. In fact, for global $\mathrm{U}_{1,2}$ only the product $\mathrm{U}=\mathrm{U}_2\,\mathrm{U}_1^{-1}$ is relevant. In this way we have uncovered the symmetry U(1) $\times$ U(1) in (3.9) of the NCSG$_1$ model in the process of constructing the conserved currents of the corresponding NCMT$_1$ sector. Notice that the currents $j^{(1)\,\mu}$ and $j^{(2)\,\mu}$ differ only by a sign in the commutative limit (recall the bosonic nature of the matter fields); not so on NC euclidean space.

Moreover, a copy of the connection (5.15)–(5.16) with the changes $\widetilde{W}^{\pm}\leftrightarrow W^{\pm}$ together with a corresponding zero-curvature equation reproduces the other set of equations (5.9)–(5.10).

JHEP03(2005)037

Therefore, one can conclude that the Lechtenfeld et al. $NCSG_1$ model for two real fields ($\varphi_\pm$) of section 3.1 corresponds to the $NCMT_1$ theory defined in (5.1) for two types of matter fields $\widetilde{\psi}$ and ψ. In section 6 we discuss this correspondence on the level of the solitonic solutions of the $NCATM_1$ model.

In the $NCSG_1$ sector one must have a topological charge corresponding to the above Noether charge. The physical scalar fields associated to this topological charge may be correctly identified in the commutative limit $\theta \to 0$ of the $NCSG_1$ eqs. of motion (3.14)–(3.15). In fact, the combination $(\varphi_+ + \varphi_-)$ carries the charge in the $NCSG_1$ sector as defined in (3.16). This correspondence can be better understood in the context of the decoupling sectors of the $NCATM_1$ model such that the one-soliton solution satisfies the Noether and topological currents equivalence (B.13) also in the NC case (see below).

Let us disclose some comments on the $NCMT_1$ action written for Dirac fermions. The $NCMT_1$ lagrangian (5.6) contains the non-standard interaction term $j^{(2)}_\mu \star j^{(2)\,\mu} \sim \psi_R \star \widetilde{\psi}_R \star \psi_L \star \widetilde{\psi}_L$, which, to our knowledge, has not been considered previously in the literature. The bosonization process of the NC extension of the usual Thirring interaction performed in [27, 28] considers only the interaction term $j^{(1)}_\mu \star j^{(1)\,\mu}$. The NC version of the MT model with $j^{(1)}_\mu \star j^{(1)\,\mu}$ interaction corresponds to a bosonized theory whose dynamics is governed by a NC WZW action plus a bosonized Thirring coupling and a NC cosine potential [28], then the bosonized model resembles one of the sectors, say g sector, of the Grisaru-Penati model (3.18). It is expected that the bosonization procedure of the $NCMT_1$ model will provide a bosonic action of the Lechtenfeld et al. type model (3.11). On the other hand, gauge theories with fermions in the bi-fundamental representation (5.21) in NC Euclidean space have been considered in [26] in order to study chiral anomalies in the NC context.

5.2 (B)NCMT$_2$

As mentioned in the last paragraph of section 4 we expect that another $NCMT_2$ version will appear when one performs similar decoupling processes for the extended system with $\{F^\pm, W^\pm\}$ and $\{\mathcal{F}^\pm, \mathcal{W}^\pm\}$ fields. In fact, a copy of the $NCMT_1$ action (5.1), as well as the relevant zero-curvature equation of motion can be written for the fields $\{\mathcal{F}^\pm, \mathcal{W}^\pm\}$.

For $\mathcal{W}^\pm$, $\widetilde{\mathcal{W}}^\pm$ in components we shall assume to contain the fields of type (Ψ, $\widetilde{\Psi}$)

$$\mathcal{W}^+ = \sqrt{\frac{4i}{m_\Psi}}\Big(\Psi_L E^0_+ + \widetilde{\Psi}_L E^1_-\Big), \quad \mathcal{W}^- = -\sqrt{\frac{4i}{m_\Psi}}\Big(\Psi_R E^{-1}_+ - \widetilde{\Psi}_R E^0_-\Big) \tag{5.22}$$

$$\Big[E_{-2}\,, \widetilde{\mathcal{W}}^+\Big] = -\sqrt{\frac{im_\Psi}{4}}\Big(\Psi_L E^{-1}_+ - \widetilde{\Psi}_L E^0_-\Big); \quad \Big[E_2\,, \widetilde{\mathcal{W}}^-\Big] = \sqrt{\frac{im_\Psi}{4}}\Big(\Psi_R E^0_+ + \widetilde{\Psi}_R E^1_-\Big). \tag{5.23}$$

Thus, one can write the (B)NCMT$_2$ action for 4 types of matter fields $\widetilde{\psi}\,,\psi\,,\widetilde{\Psi}$ and Ψ as

$$
\begin{aligned}
S_{NCMT_2}[W^{\pm},\widetilde{W}^{\pm},\mathcal{W}^{\pm},\widetilde{\mathcal{W}}^{\pm}] &\equiv S[W^{\pm},\widetilde{W}^{\pm}]+S[\mathcal{W}^{\pm},\widetilde{\mathcal{W}}^{\pm}]\\
&= \int d^2x\Big\{\Big[2i\widetilde{\psi}_L\partial_+\psi_L+2i\widetilde{\psi}_R\partial_-\psi_R-im_{\psi}\Big(\widetilde{\psi}_R\psi_L-\widetilde{\psi}_L\psi_R\Big)-\\
&\qquad -\lambda\Big(\widetilde{\psi}_R\star\psi_R\star\widetilde{\psi}_L\star\psi_L+\psi_R\star\widetilde{\psi}_R\star\psi_L\star\widetilde{\psi}_L\Big)\Big]+\\
&\qquad +\Big[\psi_{R,L}\rightarrow\Psi_{R,L},\widetilde{\psi}_{R,L}\rightarrow\widetilde{\Psi}_{R,L}\Big]\Big\} \qquad (5.24)
\end{aligned}
$$

which is related to the Grisaru-Penati model (NCSG$_2$) defined for U$(1)_C$ fields g and $\bar{g}$ of section 3.2.

The eqs. of motion comprise (5.11)–(5.14) for $\psi,\widetilde{\psi}$ and analogous eqs. for $\Psi,\widetilde{\Psi}$.

In addition to the eqs. (5.18)–(5.19) for the fields $\{\psi,\,\widetilde{\psi}\}$, we must have the equations

$$
\begin{aligned}
\partial_-(\widetilde{\Psi}_R\star\Psi_R)-\partial_+(\widetilde{\Psi}_L\star\Psi_L) \;=\;& m_{\Psi}(\widetilde{\Psi}_R\star\Psi_L+\widetilde{\Psi}_L\star\Psi_R)-\\
&-i\,\lambda(\widetilde{\Psi}_R\star\Psi_R\star\widetilde{\Psi}_L\star\Psi_L-\widetilde{\Psi}_L\star\Psi_L\star\widetilde{\Psi}_R\star\Psi_R) \qquad (5.25)
\end{aligned}
$$

and an additional equation obtained from (5.25) by conveniently substituting

$$\{\Psi_R,\,\Psi_L\}\leftrightarrow\{\widetilde{\Psi}_R,\,\widetilde{\Psi}_L\} \qquad \text{and} \qquad i\rightarrow -i\,. \qquad (5.26)$$

The equations (5.25) and the one obtained through (5.26) are also satisfied as the result of the relevant field equations.

The currents conservation laws become (5.20) for $(\psi,\widetilde{\psi})$ and analogous ones for $\Psi,\widetilde{\Psi}$, i.e.

$$\partial_-(\widetilde{\Psi}_R\star\Psi_R)+\partial_+(\widetilde{\Psi}_L\star\Psi_L)=0\,; \qquad \partial_-(\Psi_R\star\widetilde{\Psi}_R)+\partial_+(\Psi_L\star\widetilde{\Psi}_L)\;=\;0\,. \qquad (5.27)$$

However, the symmetries associated to the currents (5.20) and (5.27) must be discussed in a correct way taking into account the complexified U$(1)_C$ symmetries of its NCSG$_2$ sector (3.17). Thus, as in the above discussion on the $NCSG_1\leftrightarrow NCMT_1$ case, we may attempt to recognize the U$(1)_C$ group symmetries (3.17) of the NCSG$_2$ model in the corresponding NCMT$_2$ theory. As mentioned above in the NC case there are various ways to perform the Noether procedure in order to obtain the currents [25]. The U$(1)_C$ symmetry of a free massless fermion in commutative space has been considered in [20]. Following the above discussions on the implementation of a global symmetry in the NC case the most general U$(1)_C$ transformations deserve attention

$$
\begin{aligned}
\psi_R &\rightarrow h_R\star\psi_R\star g_R^{-1}; \qquad \psi_L\rightarrow h_L\star\psi_L\star g_L^{-1}, \qquad (5.28)\\
\widetilde{\psi}_R &\rightarrow g_R\star\widetilde{\psi}_R\star h_R^{-1}; \qquad \widetilde{\psi}_L\rightarrow g_L\star\widetilde{\psi}_L\star h_L^{-1}. \qquad (5.29)
\end{aligned}
$$

In (5.28)–(5.29) one has

$$
\begin{aligned}
h_R &= e^{[i\lambda(x)-\rho(x)]};\quad h_L=e^{[i\lambda(x)+\rho(x)]};\quad h_R^{\dagger}=h_L^{-1},\quad \lambda,\rho=\text{real functions} \qquad (5.30)\\
g_R &= e^{[i\sigma(x)-\zeta(x)]};\quad g_L=e^{[i\sigma(x)+\zeta(x)]};\quad g_R^{\dagger}=g_L^{-1},\quad \sigma,\zeta=\text{real functions}\,. \qquad (5.31)
\end{aligned}
$$

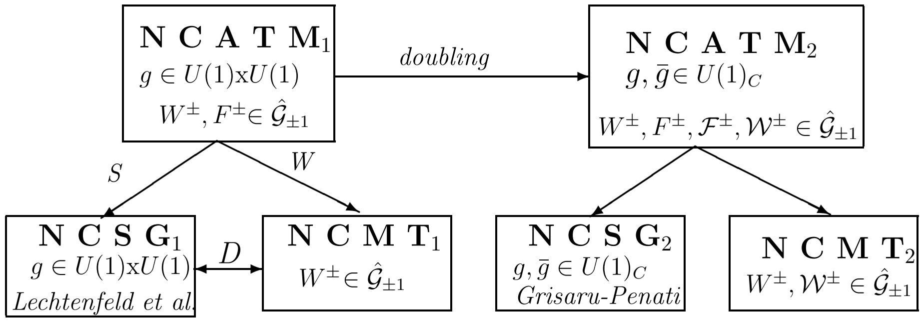

Figure 1: The models and their relationships, as well as the field contents. The duality: S=strong sector; W=weak sector; D= S-W duality

In the first equation of (5.20) one recognizes the current associated to the unitary U(1) sector of the groups g_R, g_L in (5.28)–(5.29); i.e. the $\mathrm{U}(1)_C$ symmetry provided that $g_R = g_L$ ($\zeta = 0$). On the other hand, the second equation of (5.20) corresponds to the other U(1) symmetry related to the groups h_R, h_L in (5.28)–(5.29) such that $h_R = h_L$ ($\rho = 0$) . Then the non-unitary representations of the star-localized U(1) symmetries (i.e. a representation of $\mathrm{U}(1)_C/\,\mathrm{U}(1)$) do not provide conserved currents through the Noether procedure in the NCMT_2 theory. This fact is clearly observed in the mass term of (5.24) which is not invariant under global $\mathrm{U}(1)_C$ symmetries given by $h_R g_R^{-1}$ and $h_L g_L^{-1}$. Moreover, the interaction terms are invariant under the global symmetries but not under the localized symmetries.

Similar transformation rules can be associated to the fields Ψ, $\widetilde{\Psi}$ which give rise to the conservation laws in (5.27) corresponding to the other unitary subgroup of the NC symmetry group $\mathrm{U}(1)_C$ to which $\bar{g}$ is related in the NCSG_2 model.

It is clear that the sector defined by the fields ψ, $\widetilde{\psi}$ is associated to the U(1) subgroup of $\mathrm{U}(1)_C$, and the Ψ, $\widetilde{\Psi}$ sector to a U(1) subgroup of the other $\mathrm{U}(1)_C$ symmetry. Then, the full $\mathrm{U}(1)_C$ symmetries of the NCSG_2 model do not provide conserved Noether currents in the (B)NCMT_2 sector through the process of star localizing the symmetries.

The zero-curvature condition encodes integrability even in the NC extension of integrable models [5], then we may conclude that the (Bosonic) $\mathrm{NCMT}_{1,2}$ theories are integrable and infinite conserved charges may be donstructed for them.

In figure 1 we have outlined the various relationships. Notice that we emphasized the duality relationship $\mathrm{NCSG}_1 \leftrightarrow \mathrm{NCMT}_1$ since in this case the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of the NCSG_1 model is implemented in the star-localized Noether procedure to get the U(1) currents of the NCMT_1 sector.

6. Non-commutative solitons and strong-weak duality

In this section we will show that the $\mathrm{NCATM}_{1,2}$ models reduce to the ordinary ATM theory in the commutative limit $\theta \to 0$. Moreover, we will deal with the problem of the soliton-particle and strong-weak mappings by explicitly constructing the non-commutative solitons of the $\mathrm{NCATM}_{1,2}$ models, respectively. In the following all field products are understood to be $*$ products.

6.1 NCATM$_2$ model and U$(1)_C$ parameterization

Let us consider first the second model and the $F^{\pm}$ fields defined by (B.5) in the matrix representation (A.1) and g given in (3.17). Then the equation (2.3) becomes

$$\frac{-i}{m_\psi}\begin{pmatrix} \partial_-(e^{-i\varphi}\partial_+ e^{i\varphi}) & 0 \\ 0 & \partial_-(e^{i\varphi}\partial_+ e^{-i\varphi}) \end{pmatrix} =$$
$$= \begin{pmatrix} \psi_L e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} + e^{i\varphi}\psi_R e^{i\varphi}\widetilde{\psi}_L & 0 \\ 0 & -\widetilde{\psi}_L e^{i\varphi}\psi_R e^{i\varphi} - e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi}\psi_L \end{pmatrix}. \quad (6.1)$$

Taking the trace of (6.1) one can get the equation

$$\frac{-i}{m_\psi}\partial_-\left(e^{-i\varphi}\partial_+ e^{i\varphi} + e^{i\varphi}\partial_+ e^{-i\varphi}\right) =$$
$$= \psi_L e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} + e^{i\varphi}\psi_R e^{i\varphi}\widetilde{\psi}_L - \widetilde{\psi}_L e^{i\varphi}\psi_R e^{i\varphi} - e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi}\psi_L\,.$$

This equation is related to the U$(1)_V$ subgroup of the group GL(2) related to the construction of NCATM$_2$, and reduces to a trivial equation in the limit $\theta \to 0$.

Replacing (B.4)–(B.5) into the equations (2.6) one can get the following system of equations

$$\partial_+\psi_L = -\frac{m_\psi}{2}e^{i\varphi}\psi_R e^{i\varphi}\,, \qquad \partial_+\widetilde{\psi}_L = -\frac{m_\psi}{2}e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} \quad (6.2)$$
$$\partial_-\psi_R = \frac{m_\psi}{2}e^{-i\varphi}\psi_L e^{-i\varphi}\,, \qquad \partial_-\widetilde{\psi}_R = \frac{m_\psi}{2}e^{i\varphi}\widetilde{\psi}_L e^{i\varphi}\,. \quad (6.3)$$

Notice that the equations (6.1)–(6.3) reduce to the ordinary ATM equations (B.6)–(B.8) in the limit $\theta \to 0$.

The decoupling equations (4.1)–(4.4) provide some relationships between the fields of the weak and strong sectors of the NCATM$_2$ model. Thus, from (4.1)) one gets

$$\begin{pmatrix} \psi_L e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} + e^{i\varphi}\psi_R e^{i\varphi}\widetilde{\psi}_L & 0 \\ 0 & -\widetilde{\psi}_L e^{i\varphi}\psi_R e^{i\varphi} - e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi}\psi_L \end{pmatrix} =$$
$$= \frac{iM^2}{m_\psi}\begin{pmatrix} e^{2i\varphi} - e^{-2i\varphi} & 0 \\ 0 & e^{-2i\varphi} - e^{2i\varphi} \end{pmatrix} \quad (6.4)$$

Taking the trace of (6.4) one can get the equation

$$\psi_L e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} + e^{i\varphi}\psi_R e^{i\varphi}\widetilde{\psi}_L - \widetilde{\psi}_L e^{i\varphi}\psi_R e^{i\varphi} - e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi}\psi_L = 0\,. \quad (6.5)$$

This equation reduces to a trivial equation in the limit $\theta \to 0$.

The remaining eqs. (4.2)–(4.3) give

$$m_\psi e^{-i\varphi}\psi_L e^{-i\varphi} = m_\psi\psi_L - i\lambda\left(\psi_R\widetilde{\psi}_L\psi_L + \psi_L\widetilde{\psi}_L\psi_R\right) \quad (6.6)$$
$$m_\psi e^{i\varphi}\widetilde{\psi}_L e^{i\varphi} = m_\psi\widetilde{\psi}_L + i\lambda\left(\widetilde{\psi}_R\psi_L\widetilde{\psi}_L + \widetilde{\psi}_L\psi_L\widetilde{\psi}_R\right) \quad (6.7)$$
$$m_\psi e^{i\varphi}\psi_R e^{i\varphi} = m_\psi\psi_R + i\lambda\left(\psi_L\widetilde{\psi}_R\psi_R + \psi_R\widetilde{\psi}_R\psi_L\right) \quad (6.8)$$
$$m_\psi e^{-i\varphi}\widetilde{\psi}_R e^{-i\varphi} = m_\psi\widetilde{\psi}_R - i\lambda\left(\widetilde{\psi}_L\psi_R\widetilde{\psi}_R + \widetilde{\psi}_R\psi_R\widetilde{\psi}_L\right). \quad (6.9)$$

Moreover, the eqs. (6.4)–(6.9) should be satisfied by a subset of solutions of the field equations (6.1)–(6.3) such that the two sectors decouple.

6.1.1 NCATM$_2$ one-solitons

It is known that the one-soliton solution of certain models solves their NC counterparts. This feature holds in models such as the sine-Gordon (SG) and nonlinear Schrodinger(NS) and their NC versions NCSG$_2$ [3, 5] and NCNS [29], respectively. We will show that this behavior is maintained in the ATM and its corresponding NCATM$_2$ model. This would mean that from the point of view of one-soliton solutions the ATM and the NCATM$_2$ equations of motion are the same. Of course, the constraint (6.2) must also be verified for the decoupling sectors to have one-soliton solutions. The study of multi-solitons is also interesting, e.g. in order to check the validity of the NC version of the Noether and topological currents equivalence (B.13).

It is also known that if $f(x_0, x_1)$ and $g(x_0, x_1)$ depend only on $(x_1 - vx_0)$, then the product $f \star g$ coincides with the ordinary product $f.g$ [5, 29]. Therefore, all the $\star$ products in (6.1)–(6.3) for this type of functions become the same as the ordinary ones, in this way our system of Eqs. reduce to the usual ATM Eqs. of motion (B.6)–(B.8); observe that e.g., $e_\star^{i\varphi} = e^{i\varphi}$, and the products of type, $e_\star^{i\varphi} \star \psi_R \star e_\star^{i\varphi} \star \widetilde{\psi}_L = e^{2i\varphi}\psi_R\widetilde{\psi}_L$.

The best example of that is the real one-soliton solution of (B.6)–(B.8) which was calculated in [14] and it is given by

$$\varphi = 2\arctan\left(\exp\left(\frac{2m_\psi\left(x - x_0 - vt\right)}{\sqrt{1-v^2}}\right)\right) \tag{6.10}$$

$$\psi = e^{i\theta}\sqrt{m_\psi}\, e^{m_\psi\left(x-x_0-vt\right)/\sqrt{1-v^2}} \begin{pmatrix} \left(\frac{1-v}{1+v}\right)^{1/4} \dfrac{1}{1+ie^{2m_\psi\left(x-x_0-vt\right)/\sqrt{1-v^2}}} \\ -\left(\frac{1+v}{1-v}\right)^{1/4} \dfrac{1}{1-ie^{2m_\psi\left(x-x_0-vt\right)/\sqrt{1-v^2}}} \end{pmatrix} \tag{6.11}$$

and the solution for $\widetilde{\psi}$ is the complex conjugate of ψ. Thus, (6.10)–(6.11) is a *one-soliton* solution of NCATM$_2$. Notice that this solution corresponds to a subset of solutions such that $g \in \mathrm{U}(1)$ providing the ATM model with real lagrangian.

It is a simple observation that the solution (6.10)–(6.11) satisfies the relationships (B.12)–(B.13) between the matter field bilinears and the Toda field.

Notice that for this solution the decoupling eq. (6.5) becomes trivial and in eqs. (6.6)–(6.9) one can drop out an overall factor $\psi_{R,L}$(or $\widetilde{\psi}_{R,L}$) in each equation. Thus, all of the decoupling equations turn out to be written in terms of fields of type $e^{\pm 2i\varphi}$ and bilinears $\widetilde{\psi}_R\psi_L, \widetilde{\psi}_L\psi_R$. Then, a relation between the NCSG$_2$ parameter M and the NCMT$_2$ coupling λ can be established by using the decoupling equations (6.4)–(6.9) and the relationship (B.12). The matter field bilinears expressed in terms of exponentials of the Toda field obtained in this way for *one-soliton (anti-soliton)* when conveniently substituted into (6.4)–(6.9) provide a relationship between the parameters M and λ

$$\lambda = -\frac{m_\psi^2}{2M^2}\,. \tag{6.12}$$

JHEP03(2005)037

Therefore, one can say that for this parameter relationship the one-soliton solution (6.10)–(6.11) satisfies the decoupling equations (6.4)–(6.9). This relationship is the same as the one given in [7] in the ordinary case. This shows that the strong-weak mapping holds also in the NC case.

The topological charge in the NC case may be defined by

$$Q_{\text{topol.}} = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx^1 \frac{d\varphi}{dx^1} \equiv \sum_n \theta^n Q^{(n)} , \tag{6.13}$$

where φ is the localized "N-soliton" solution to all orders in θ. Notice that, from (6.13), one indeed has $Q_{\text{topol.}} = 1$ for the solution (6.10)–(6.11). In particular for the one-soliton, since this solution is exact (without expansion in θ) for the NC version, the higher order terms in (6.13) vanish.

6.2 NCATM$_1$ model and U(1) $\times$ U(1) parameterization

Let us consider $E_{\pm 2}$ given in (B.4), the $F^{\pm}$ fields defined by (B.5) in the matrix representation (A.1) and the U(1) $\times$ U(1) parameterization for g given in (3.9). Then the eq. (2.3) becomes

$$\begin{aligned} \frac{-i}{m_\psi} \begin{pmatrix} \partial_-(e^{-i\varphi_+}\partial_+ e^{i\varphi_+}) & 0 \\ 0 & \partial_-(e^{i\varphi_-}\partial_+ e^{-i\varphi_-}) \end{pmatrix} = \\ = \begin{pmatrix} \psi_L e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} + e^{i\varphi_+}\psi_R e^{i\varphi_-}\widetilde{\psi}_L & 0 \\ 0 & -\widetilde{\psi}_L e^{i\varphi_+}\psi_R e^{i\varphi_-} - e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+}\psi_L \end{pmatrix}. \end{aligned} \tag{6.14}$$

Taking the trace of (6.14) one can get the equation

$$\begin{aligned} \frac{-i}{m_\psi}\partial_-(e^{-i\varphi_+}\partial_+ e^{i\varphi_+} + e^{i\varphi_-}\partial_+ e^{-i\varphi_-}) = \\ = \psi_L e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} + e^{i\varphi_+}\psi_R e^{i\varphi_-}\widetilde{\psi}_L - \widetilde{\psi}_L e^{i\varphi_+}\psi_R e^{i\varphi_-} - e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+}\psi_L . \end{aligned} \tag{6.15}$$

This equation is related to the U(1)$\times$U(1) sector of the group GL(2) used to construct the NCATM$_1$ model, and in the commutative limit reduces to the U(1)$_V$ free field equation of motion

$$\partial^2(\varphi_+ - \varphi_-) = 0 . \tag{6.16}$$

The equations (2.6) for this parameterization provide the system

$$\partial_+\psi_L = -\frac{m_\psi}{2} e^{i\varphi_+}\psi_R e^{i\varphi_-} , \qquad \partial_+\widetilde{\psi}_L = -\frac{m_\psi}{2} e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} \tag{6.17}$$

$$\partial_-\psi_R = \frac{m_\psi}{2} e^{-i\varphi_+}\psi_L e^{-i\varphi_-} , \qquad \partial_-\widetilde{\psi}_R = \frac{m_\psi}{2} e^{i\varphi_-}\widetilde{\psi}_L e^{i\varphi_+} . \tag{6.18}$$

Subtracting the diagonal elements of the matrix eq. (6.14) in the *commutative* limit one gets

$$\partial_+\partial_-(\varphi_+ + \varphi_-) = 2m_\psi \Big(\psi_L \widetilde{\psi}_R e^{-i(\varphi_+ + \varphi_-)} + \psi_R \widetilde{\psi}_L e^{i(\varphi_+ + \varphi_-)} \Big). \tag{6.19}$$

JHEP03(2005)037

In view of the commutative limits (6.16) and (6.19) one can conclude that in this limit the system of equations for the NCATM_1 model (6.14)–(6.18) reduce to the ordinary ATM model equations of motion (B.6)–(B.8) if one identifies

$$(\varphi_+ + \varphi_-) \equiv 2\varphi\,. \tag{6.20}$$

In this limit the free field $(\varphi_+ - \varphi_-)$ (6.16) completely decouples from the ordinary ATM model defined for the $\mathrm{U}(1)_A$ field φ and the higher grading $\psi, \widetilde{\psi}$ fields.

The decoupling equations (4.1)–(4.4) provide some relationships between the fields of the weak-strong sectors of NCATM_1. Thus, from (4.1)) one gets

$$\begin{aligned}
&\begin{pmatrix} \psi_L e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} + e^{i\varphi_+}\psi_R e^{i\varphi_-}\widetilde{\psi}_L & 0 \\ 0 & -\widetilde{\psi}_L e^{i\varphi_+}\psi_R e^{i\varphi_-} - e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+}\psi_L \end{pmatrix} = \\
&\qquad = \frac{iM^2}{m_\psi}\begin{pmatrix} e^{i\varphi_+}e^{i\varphi_-} - e^{-i\varphi_-}e^{-i\varphi_+} & 0 \\ 0 & e^{-i\varphi_-}e^{-i\varphi_+} - e^{i\varphi_+}e^{i\varphi_-} \end{pmatrix}.
\end{aligned} \tag{6.21}$$

Taking the trace of (6.21) one can get the equation

$$\psi_L e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} + e^{i\varphi_+}\psi_R e^{i\varphi_-}\widetilde{\psi}_L - \widetilde{\psi}_L e^{i\varphi_+}\psi_R e^{i\varphi_-} - e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+}\psi_L = 0\,. \tag{6.22}$$

This equation reduces to a trivial equation in the limit $\theta \to 0$.

The remaining eqs. (4.2)–(4.3) give

$$m_\psi e^{-i\varphi_+}\psi_L e^{-i\varphi_-} = m_\psi \psi_L - i\lambda\left(\psi_R\widetilde{\psi}_L\psi_L + \psi_L\widetilde{\psi}_L\psi_R\right) \tag{6.23}$$

$$m_\psi e^{i\varphi_-}\widetilde{\psi}_L e^{i\varphi_+} = m_\psi \widetilde{\psi}_L + i\lambda\left(\widetilde{\psi}_R\psi_L\widetilde{\psi}_L + \widetilde{\psi}_L\psi_L\widetilde{\psi}_R\right) \tag{6.24}$$

$$m_\psi e^{i\varphi_+}\psi_R e^{i\varphi_-} = m_\psi \psi_R + i\lambda\left(\psi_L\widetilde{\psi}_R\psi_R + \psi_R\widetilde{\psi}_R\psi_L\right) \tag{6.25}$$

$$m_\psi e^{-i\varphi_-}\widetilde{\psi}_R e^{-i\varphi_+} = m_\psi \widetilde{\psi}_R - i\lambda\left(\widetilde{\psi}_L\psi_R\widetilde{\psi}_R + \widetilde{\psi}_R\psi_R\widetilde{\psi}_L\right). \tag{6.26}$$

The eqs. (6.21)–(6.26) should be satisfied by a subset of solutions of the field equations (6.14)–(6.15) and (6.17)–(6.18) such that the weak-strong sectors of NCATM_1 decouple.

6.2.1 NCATM_1 one-solitons

As usual, to search for one-soliton solutions we assume all the fields depend on $(x^1 - vx^0)$, then from (6.14)–(6.15) one can arrive at the eqs. (6.16) and (6.19). Notice that for this type of solutions one can formally remove the $\star$'s in all the expressions. Therefore, (6.17)–(6.18) reduce to the corresponding equations of the ordinary ATM model (B.7)–(B.8) for the identification (6.20).

The eq. (6.16) admits the trivial solution $\varphi_+ - \varphi_- = 0$, then in view of the identification (6.20) one can write $\varphi_+ = \varphi_- = \varphi$.

Moreover, in this limit the decoupling eqs. (6.21)–(6.26) reduce to the eqs. (6.4)–(6.9), respectively. Then, the *one-soliton* (6.10)–(6.11) solves the system (6.14)–(6.15) and (6.17)–(6.18) for $\varphi_+ = \varphi_- = \varphi$. In fact, for this identification basically all the discussions regarding the one-soliton solution of section 6.1.1 are valid.

Therefore one can conclude that the weak-strong relationship (6.12) also holds in the soliton sector of the NCATM_1 model.

The N-solitons of the NCSG_1 model have been proposed in [6]. In the one-soliton case it reduces to the ordinary SG soliton. Let us emphasize that the NCATM one-soliton is 1/2 the NCSG_1 soliton ($\varphi_{SG}^{1-sol} = 2\varphi_{ATM}^{1-sol} \equiv 2\varphi^{1-sol}$), this is plausible since the ATM φ^{1-sol} soliton is interacting with the matter fields ψ and $\widetilde{\psi}$, and only in the reduced model NCSG_1 one must envisage the true SG (anti-soliton)one-soliton with topological numbers ± 1, provided that one considers the usual 2π normalization in front of the integral Eq. (3.16).

7. Conclusions and discussions

Some properties of the NC extensions of the ATM model and their weak-strong phases described by $\mathrm{NCMT}_{1,2}$ and $\mathrm{NCSG}_{1,2}$ models, respectively, have been considered. The figure 1 summarizes the main relationships, as well as the field contents in each model. In the process of constructing the Noether currents one recognizes the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry in both $\mathrm{NCMT}_{1,2}$ models (as a subgroup of $\mathrm{U}(1)_C \times \mathrm{U}(1)_C$ in the model 2). In the $\theta \to 0$ limit we have the following: $\mathrm{NCATM}_{1,2} \to$ ATM; $\mathrm{NCSG}_{1,2}$ (the real sector of model 2) $\to$ SG(plus a free scalar in the case of model 1) and $\mathrm{NCMT}_{1,2}$ (one of the sectors of the model 2) $\to$ MT. The main result is the classical mappings $\mathrm{NCSG}_{1,2} \leftrightarrow \mathrm{NCMT}_{1,2}$, respectively (the real soliton sector of the models 2). The mapping relating the models 1 is more promising since it is expected to hold on the quantum level in view of the tree level results of [6], regarding the nice properties in the NCSG_1 sector, such as, factorizable and causal S-matrix. This fact seems to be related to the fact that the whole $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetries of the NCSG_1 model appear in the localization procedure to get the Noether currents in its NCMT_1 dual, whereas the $\mathrm{U}(1)_C$ symmetry of NCSG_2 is not recognized in this process when the NCMT_2 model is considered since the mass terms and the interaction terms are not invariant under the non-unitary sector $\mathrm{U}(1)_C/\,\mathrm{U}(1)$.

The same one-soliton solution solves both $\mathrm{NCATM}_{1,2}$ models (the real sector of model 2) and the fact that this solution depends only on $(x - vt)$ allowed us to reduce the problem basically on that of the known ATM properties, in this way establishing the weak-strong correspondence in the NC realm.

It would be interesting to understand what actually determines the systems to be integrable (in the sense of possessing a factorizable and causal S-matrix) and dual to each other. A hint in this direction, for the systems $\mathrm{NCSG}_1 \leftrightarrow \mathrm{NCMT}_1$, seems to be that the models originate directly through a reduction processes starting from the NC WZNW type action for the Toda field coupled to the higher grading matter fields (NCATM_1), such that the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry is relevant, both in the construction of the NCSG_1 model and in the star-localized Noether procedure to construct the $U_{1,2}(1)$ currents in the NCMT_1 sector. Recall that the commutative integrable ATM models are themselves derivable from the (two-loop) WZNW model [7, 14] when a corresponding hamiltonian reduction is performed [30].

Various aspects of the models studied above deserve attention in future research, e.g., the NC multi-solitons of the NCATM$_{1,2}$ models, the bosonization of the NCMT$_{1,2}$ models and its multi-fermion extensions [13], the NC zero-curvature formulation of the MT model defined for grassmannian fields. In particular, in the bosonization process of NCMT$_{1,2}$ models and their multi-fermion extensions, initiated in [28] by directly starring the usual Thirring interaction, we believe that a careful understanding of the star-localized NC Noether symmetries, as well as the classical soliton spectrum would be desirable.

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A. Affine Lie algebra $\hat{sl}(2)$

The Lie algebra $sl(2)$ is formed by all 2×2 complex matrices with zero trace. Let us assume the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = (E_+)^T . \tag{A.1}$$

We use the invariant bilinear form on $sl(2)$ defined by

$$\langle x \, , \, y \rangle \equiv \langle x \, y \rangle = \mathrm{Tr}(xy) \, , \qquad x, y \in sl(2) \, . \tag{A.2}$$

The affine Kac-Moody algebra $\widehat{sl}(2)$ is constructed as follows. Consider the loop algebra

$$\mathcal{L}(sl(2)) = \mathcal{C}[\zeta, \zeta^{-1}] \otimes sl(2) \, , \tag{A.3}$$

where $\mathcal{C}[\zeta, \zeta^{-1}]$ is the algebra of Laurent polynomials in ζ. An element of $\mathcal{L}(sl(2))$ is a finite linear combination of the elements of the form $\zeta^m \otimes x$, where $m \in \mathbb{Z}$ and $x \in sl(2)$. The structure of a Lie algebra in $\mathcal{L}(sl(2))$ is introduced by the relation

$$[\, \zeta^m \otimes x \, , \, \zeta^n \otimes y \,] = \zeta^{m+n} \otimes [\, x \, , \, y \,] \, . \tag{A.4}$$

The elements of the form $1 \otimes x$, $x \in sl(2)$ are identified with the Lie algebra $sl(2)$. Thus the algebra $sl(2)$ is a subalgebra of $\mathcal{L}(sl(2))$. This allows us to write $\zeta^m \otimes x$ in the form $\zeta^m x$.

Let us denote by $\tilde{\mathcal{L}}(sl(2))$ the algebra extended by the one dimensional center operator C. The commutation relations in the Lie algebra $\tilde{\mathcal{L}}(sl(2)) = \mathcal{L}(sl(2)) \oplus \mathcal{C}\, C$ are given by

$$[H^m \, , \, H^n] = 2\, m \, C \, \delta_{m+n,0} \, , \tag{A.5}$$

$$[H^m \, , \, E_\pm^n] = \pm 2 \, E_\pm^{m+n} \, , \tag{A.6}$$

$$[E_+^m \, , \, E_-^n] = H^{m+n} + m \, C \, \delta_{m+n,0} \, , \tag{A.7}$$

where the elements $H^m = \zeta^m H$, $E_\pm^m = \zeta^m E_\pm$ have been defined.

JHEP03(2005)037

Next, denote by $\widehat{sl}(2)$ the Lie algebra which is obtained by adjoining to $\tilde{\mathcal{L}}(sl(2))$ a derivation operator $D = \zeta(\mathrm{d}/\mathrm{d}\zeta)$. The commutation relations for the Lie algebra $\widehat{sl}(2)$ are defined by relations (A.5)–(A.7) and by the equalities

$$[\,D\,,\,\zeta^m x\,] = m\,\zeta^m x\,, \qquad [\,D\,,\,C\,] = 0\,. \tag{A.8}$$

The generator $Q \equiv \frac{1}{2}H^0 + 2D$ is the principal gradation operator [31]. Then its eigensubspaces are

$$\begin{aligned} \hat{\mathcal{G}}_0 &= \{H^0, C, Q\}\,; \\ \hat{\mathcal{G}}_{2n+1} &= \{E_+^n, E_-^{n+1}\} \qquad n \in \mathbb{Z}\,; \\ \hat{\mathcal{G}}_{2n} &= \{H^n\}, \qquad n \in \{\mathbb{Z} - 0\}\,. \end{aligned} \tag{A.9}$$

B. The commutative $sl(2)$ ATM model

The $sl(2)$ ATM theory has been studied from different points of view [7, 10, 12, 8]. Here we present some results expressed in matrix form which are relevant to our discussions. The theory contains the usual sine-Gordon (SG) and the massive Thirring (MT) models describing the soliton/particle correspondence of its spectrum [7, 10].

The equations of motion of the $\hat{sl}(2)$ conformal affine Toda model coupled to matter (CATM) are given by

$$\partial_-(g^{-1}\partial_+ g) + \partial_-\partial_+\nu C = e^{\eta}\Big[F^-\,,\,gF^+g^{-1}\Big] \tag{B.1}$$

$$\partial_+ F^- = -e^{\eta}[E_{-2}\,,\,gF^+g^{-1}]\,, \qquad \partial_- F^+ = e^{\eta}[E_2\,,\,g^{-1}F^-g]\,, \tag{B.2}$$

$$\partial_-\partial_+\eta Q = 0\,, \tag{B.3}$$

where

$$E_{\pm 2} = \frac{m_\psi}{4}H^{\pm 1}\,, \qquad g = e^{i\varphi H^0}\,, \tag{B.4}$$

$$F^+ = \sqrt{im_\psi}\Big(\psi_R E_+^0 + \widetilde{\psi}_R E_-^1\Big), \qquad F^- = \sqrt{im_\psi}\Big(\psi_L E_+^{-1} - \widetilde{\psi}_L E_-^0\Big)\,. \tag{B.5}$$

We have denoted by H^n, $E_\pm^n$, and C the Chevalley basis generators of the $\hat{sl}(2)$ affine Kac-Moody algebra and Q the principal gradation generator.

Taking into account the parameterizations (B.4)–(B.5) and by setting $\eta = 0$ one gets the off-critical $sl(2)$ ATM model equations of motion from (B.1)–(B.2)[2]

$$\partial_+\partial_-\varphi = m_\psi\Big(\psi_L\widetilde{\psi}_R e^{-2i\varphi} + \psi_R\widetilde{\psi}_L e^{2i\varphi}\Big) \tag{B.6}$$

$$\partial_+\psi_L = -\frac{m_\psi}{2}\psi_R e^{2i\varphi}\,, \qquad \partial_+\widetilde{\psi}_L = -\frac{m_\psi}{2}\widetilde{\psi}_R e^{-2i\varphi} \tag{B.7}$$

$$\partial_-\psi_R = \frac{m_\psi}{2}\psi_L e^{-2i\varphi}\,, \qquad \partial_-\widetilde{\psi}_R = \frac{m_\psi}{2}\widetilde{\psi}_L e^{2i\varphi}\,. \tag{B.8}$$

[2]Consider $\gamma_0 = -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Associated to these equations of motion one can write the ATM lagrangian[3]

$$\mathcal{L}_{ATM} = -\frac{1}{4}\partial_\mu\varphi\,\partial^\mu\varphi + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_\psi\,\bar{\psi}\,e^{2i\varphi\,\gamma_5}\,\psi\,. \tag{B.11}$$

Notice that (B.11) is a real lagrangian if φ is real and $\widetilde{\psi}$ is the complex conjugate of ψ.

The strong/weak dual phases of the model (B.11) has been uncovered by means of the symplectic and master lagrangian approaches [8, 9]. The strong phase is described by the SG model: $\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{m_\psi^2}{\lambda}\cos 2\varphi$. The usual massive Thirring model $\mathcal{L}_{MT} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m_\psi\overline{\psi}\psi + \frac{1}{2}\lambda j_\mu j^\mu$ describes the weak coupling phase. Notice the weak-strong exchange $\lambda \to \frac{1}{\lambda}$ of the coupling constant.

The one-(anti)soliton solution of the system (B.6)–(B.8) satisfies the remarkable SG and MT classical correspondence [23] in which, apart from the Noether and topological currents equivalence, MT matter field bilinears are related to the exponentials of the SG field [7].

The soliton type solutions satisfy the relationships [7]

$$\psi_R\tilde{\psi}_L = \frac{im_\psi}{2\lambda}(e^{-2i\varphi}-1)\,, \qquad \psi_L\tilde{\psi}_R = -\frac{im_\psi}{2\lambda}(e^{2i\varphi}-1)\,, \tag{B.12}$$

and the currents equivalence

$$\bar{\psi}\gamma^\mu\psi = \frac{1}{2}\epsilon^{\mu\nu}\partial_\nu\varphi\,. \tag{B.13}$$

The relationships (B.12) and (B.13) have been verified for $N = 1$ and $N = 1, 2$ solitons, respectively [7, 10].

In order to write the matrix form of the lagrangian let us consider the auxiliary fields $W^{\mp}$ defined by

$$\partial_+ W^+ = -gF^+g^{-1}\,, \qquad \partial_- W^- = -g^{-1}F^-g\,. \tag{B.14}$$

Next, setting $\eta = 0$ in (B.1)–(B.2) one can write the matrix form of the off-critical ATM action [9]

$$\begin{aligned} S[g, W^\pm, F^\pm] = I_{\mathrm{WZW}}[g] + \int\Big\{ &\frac{1}{2}\left\langle\partial_- W^-[E_2\,,\,W^-]\right\rangle - \frac{1}{2}\left\langle[E_{-2}\,,\,W^+]\partial_+ W^+\right\rangle + \\ &+\left\langle F^-\partial_+ W^+\right\rangle + \left\langle\partial_- W^- F^+\right\rangle + \\ &+\left\langle F^- g F^+ g^{-1}\right\rangle\Big\}\,, \end{aligned} \tag{B.15}$$

where $I_{\mathrm{WZW}}[g]$ is the Wess-Zumino-Witten model. The notation $\langle,\rangle$ extends the trace operation (A.2) to the affine case. Notice that since g is abelian the WZW term in I_{WZW} vanishes.

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[3] Define the Dirac fields as

$$\psi = \begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}; \qquad \widetilde{\psi} = \begin{pmatrix}\widetilde{\psi}_R\\ \widetilde{\psi}_L\end{pmatrix}; \qquad \bar{\psi} = \widetilde{\psi}^T\gamma^0\,. \tag{B.9}$$

We are using

$$x_\pm = t \pm x\,, \qquad \text{then}\,, \quad \partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)\,, \qquad \text{and} \qquad \partial^2 = \partial_t^2 - \partial_x^2 = 4\partial_-\partial_+\,. \tag{B.10}$$

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JHEP03(2005)037

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